

On Symmetrized Weight Compositions

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Abstract

A characterization of module alphabets with the Hamming weight EP (abbreviation for Extension Property) had been settled. A thoughtfully constructed piece-of-art example by J.A.Wood ([7]) finished the tour. In 2009, in [8], Frobenius bimodules were proved to satisfy the EP with respect to *symmetrized weight compositions*. In [4], the embeddability in the character group of the ambient ring R was found sufficient for a module ${}_R A$ to satisfy the EP with respect to swc built on any subgroup of $\text{Aut}_R(A)$, while the necessity remained a question.

Here, landing in a “Midway”, the necessity is proved by jumping to Hamming weight. Corollary 1.11 declares a characterization of module alphabets satisfying the EP with respect to swc.

Note: All rings are finite with unity, and all modules are finite too. This may be re-emphasized in some statements. The convention for functions is that inputs are to the left.

Symmetrized Weight Compositions

Definition 1.1. (Symmetrized Weight Compositions) Let G be a subgroup of the automorphism group of a finite R -module A . Define \sim on A by $a \sim b$ if $a = b\tau$ for some $\tau \in G$. Let A/G denote the orbit space of this action. The symmetrized weight composition is a function

$\text{swc} : A^n \times A/G \rightarrow \mathbb{Q}$ defined by,

$$\text{swc}(x, a) = \text{swc}_a(x) = |\{i : x_i \sim a\}|,$$

where $x = (x_1, \dots, x_n) \in A^n$ and $a \in A/G$.

Definition 1.2. Let G be a subgroup of $\text{Aut}_R(A)$, a map T is called a G -monomial transformation of A^n if for any $(x_1, \dots, x_n) \in A^n$

$$(x_1, \dots, x_n)T = (x_{\sigma(1)}\tau_1, \dots, x_{\sigma(n)}\tau_n),$$

where $\sigma \in S_n$ and $\tau_i \in G$ for $i = 1, \dots, n$.

Definition 1.3. (Extension Property) The alphabet A has the extension property with respect to swc if for every n , and any two linear codes $C_1, C_2 \subset A^n$, any R -linear isomorphism $f : C_1 \rightarrow C_2$ that preserves swc is extendable to a G -monomial transformation of A^n .

In [6], J.A.Wood proved that Frobenius rings do have the extension property with respect to swc. Later, in [4], it was shown that, more generally, a left R -module A has the extension property with respect to swc if it can be embedded in the character group \widehat{R} (given the standard module structure).

Theorem 1.4. (Th.4.1.3, [3]) *Let A be a finite left R -module. If A can be embedded into \widehat{R} , then A has the extension property with respect to the swc built on any subgroup G of $\text{Aut}_R(A)$. In particular, this theorem applies to Frobenius bimodules.*

We now define a new notion (the Midway!) on which we'll depend in the rest of this paper.

Definition 1.5. (Annihilator Weight) On ${}_R A$, define an equivalence relation ρ by $a\rho b$ if $\text{Ann}_a = \text{Ann}_b$, where a and b are any two elements in A and $\text{Ann}_a = \{r \in R \mid ra = 0\}$ is the annihilator of a . Clearly, Ann_a is a left ideal.

Now, on A^n we can define the annihilator weight aw that counts the number of components in each orbit.

Remark: It is easily seen that the EP with respect to Hamming weight implies the EP with respect to swc built on $\text{Aut}_R(A)$, and the EP with respect to aw as well.

Lemma 1.6. *Let ${}_R A$ be a pseudo-injective module. Then for any two elements a and b in A , $a\rho b$ if and only if $a \sim b$.*

Proof. If $a \sim b$, this means $a = b\tau$ for some $\tau \in \text{Aut}_R(A)$, and consequently $\text{Ann}_a = \text{Ann}_b$.

Conversely, if $a\rho b$, then we have (as left R -modules)

$$Ra \cong {}_R R / \text{Ann}_a = {}_R R / \text{Ann}_b \cong Rb,$$

with $ra \mapsto r + \text{Ann}_a \mapsto rb$. By Proposition 5.1 in [8], since A is pseudo-injective, the isomorphism $Ra \rightarrow Rb \subseteq A$ extends to an automorphism of A taking a to b . \square

Corollary 1.7. *If A is a pseudo-injective module, then the EP with respect to swc built on $\text{Aut}_R(A)$ is equivalent to the EP with respect to aw .*

Theorem 1.8. *If R is a chain ring, then for any pseudo-injective module A , the following are equivalent:*

1. *A has the EP with respect to swc built on $\text{Aut}_R(A)$.*
2. *A has the EP with respect to Hamming weight.*

Proof. First recall that the left ideals of a chain ring form a chain with inclusion, so we may assume that our chain is $I_0 = \text{rad} R \supset I_1 \supset I_2 \supset \dots \supset I_m \supset (0)$.

By the remark above we know that (2) implies (1). For the converse, by Corollary 1.7, it's enough to show that if A has the EP with respect to aw , then it has the EP with respect to Hamming weight. Suppose the EP with respect to aw holds, and that $f : C \rightarrow A^n$ is a monomorphism preserving Hamming weight, let $(c_1, c_2, \dots, c_n) \in C$ and $(c_1, c_2, \dots, c_n)f = (b_1, b_2, \dots, b_n)$.

Choose an element $x_0 \in I_0$ that doesn't belong to any of the smaller left ideals. Then, in the equality

$$(x_0 c_1, x_0 c_2, \dots, x_0 c_n)f = (x_0 b_1, x_0 b_2, \dots, x_0 b_n),$$

the number of annihilated components is exactly the number of those c_i 's (and b_i 's) with annihilator I_0 . The preservation of Hamming weight then gives that this number is the same on each side. Repeating this process we get that f preserves aw and hence extends to a monomial transformation. \square

Theorem 1.9. *Let R be the ring $\mathbb{M}_m(\mathbb{F}_q)$ of square matrices of size $m \times m$ over a the finite field \mathbb{F}_q , and let A be a finite R -module. Then the EP with respect to swc built on $\text{Aut}_R(A)$ holds if and only if the EP with respect to Hamming weight holds.*

Proof. The “if” part is direct. For the converse, notice that A is now injective (being a matrix module), so, again, it’s enough to prove that if A has the EP with respect to aw , then it has the EP with respect to Hamming weight.

Suppose the EP with respect to aw holds, and that $f : C \rightarrow A^n$ is a monomorphism preserving Hamming weight.

Let’s first focus on $R = \mathbb{M}_m(\mathbb{F}_q)$. R is a finite simple ring (hence semisimple and left artinian), thereby, any left ideal I has the form Re_I , where e_I is an idempotent (Theorem ix.3.7, [5]).

It is clear, then, that any left ideal I contains an element e_I that does not belong to any other left ideal not containing I . Now, if

$$(c_1, c_2, \dots, c_n)f = (b_1, b_2, \dots, b_n), \quad (1.1)$$

choose, from $c_1, c_2, \dots, c_n; b_1, b_2, \dots, b_n$, a component with a maximal annihilator I . Act on equation (1.1) by e_I , then the only zero places are those of the components in equation (1.1) with annihilator I , the preservation of Hamming weight gives the preservation of I -annihilated components. Omit these components from the list $c_1, c_2, \dots, c_n; b_1, b_2, \dots, b_n$ and choose one with the new maximal, and repeat. This gives that f preserves aw and hence extends to a monomial transformation. \square

Corollary 1.10. *Let R be the ring $\mathbb{M}_m(\mathbb{F}_q)$, and $A = \mathbb{M}_{m,k}(\mathbb{F}_q)$ where $k > m$, then A doesn’t have the EP with respect to swc.*

Proof. In [7], J.A.Wood proved that A doesn’t have the Hamming weight EP, hence, by the above A cannot have the EP with respect to swc. \square

Corollary 1.11. *If ${}_R A$ be a finite module over a finite ring R with unity, then A has the extension property with respect to swc if and only if can be embedded in \hat{R} (or equivalently A has a cyclic socle).*

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